

Using Linear Programming for Management Decisions

By Tim Wright



Linear programming creates mathematical models from real-world business problems to maximize profits, reduce costs and allocate resources.

For this business problem, executive management has asked HR to assist finance with creating a mathematical model to determine an optimal product mix that will maximize profit while also factoring labor restrictions.

In this iconic linear programming example, the company makes two products, Product A and Product B. Profit per unit is \$4 for product A and \$8 for product B. The two products have labor costs from three departments - C, D and E. Department C requires 8 hours per unit to make product A, 10 hours per unit for B and has a maximum of 11,000 man-hours to complete the entire project, department D requires 4 hours per unit for product A and 10 man-hours for B and has a maximum of 9,000 hours and department E requires 12 hours per unit for product A and 6 hours for B and has a maximum of 12,000 man-hours to produce the two products.

HR and finance's goal is to create an optimization model that maximizes profit, within the boundaries of the constraints. Constraints are the areas that limit the activities of the business. This business problem can be solved very quickly with linear programming optimization programs like Microsoft Solver, Lindo or Lingo. In this example, however, we will create the model algebraically using the graphical method. This problem is suitable for a graphical solution because there are only two decision variables, products A and B. Understanding how the graphical method works will enhance your ability to solve business problems with linear programming programs such as Solver, Lindo or Lingo.

The first step to solving the problem is to identify the variables:

Let x = the number of product A needed

Let y = the number of product B needed

The x and y denote the "decision variables."

Let z = our aim, which is to find the optimal solution from products A and B. The goal (z) is to determine the best product combination that will place the company in a profit maximizing position.

Based on the variables above, the objective function is $z = 4(x) + 8(y)$. The objective function finds the precise solution points within the entire region of possible solutions. It is called the objective function

because the solution to the business problem is some multiple of the product mix, the two decision variables. We are looking to maximize profit with the best product combination. How much of product A and how much of product B will give the company the largest profits?

Next, we list the constraints or the limitations for each department to make the two products, A & B.

Department C: $8x + 10y \leq 11,000$ (8 hours per unit to make product x, 10 hours per unit for y, with a maximum of 11,000 hours)

Department D: $4x + 10y \leq 9,000$ (4 hours per unit for x, 10 hours per unit for y, with a maximum of 9,000 hours)

Department E: $12x + 6y \leq 12,000$ (12 hours per unit for x, 6 hours per unit for y, with a maximum of 12,000 man-hours)

There's one more necessary constraint. Because we are solving this problem graphically, the model's solution will have all positive numbers. In other words, the solution will take place in the first quadrant of the graph. The fancy way of saying this is we need to add a "non-negativity constraint."

Both x and y will be positive numbers. This means x and y must be greater than zero. So,

$x \geq 0$ and $y \geq 0$ or just simply $x, y \geq 0$

Our completed mathematical model is below:

Objective function: $z = 4(x) + 8(y)$, remembering that x and y equal the *number* of products to produce.

s. t. (stands for "subject to")

The constraints:

For Department C: $8x + 10y \leq 11,000$

For Department D: $4x + 10y \leq 9,000$

For Department E: $12x + 6y \leq 12,000$

And the Non-negativity constraint: $x, y \geq 0$

The fun part begins when we plot the *constraint values* on the graph. To do this, we take a time capsule back to grade school and use Algebra 1 calculations. We first solve for x and y, then plot the coordinates.

Using Algebra steps for Department C, if $x = 0$, then $y = 1,100$. When $y = 0$, $x = 1,375$.

Using Algebra steps for Department D, if $x = 0$, then $y = 900$. When $y = 0$, $x = 2,250$.

Using Algebra steps for Department E, if $x = 0$, then $y = 2,000$. When $y = 0$, $x = 1,000$.

When plotting our graph, 2,250 is the highest unit on the x axis and 2,000 is the highest unit on the y axis. Remember, y is the vertical axis and x is the horizontal axis.

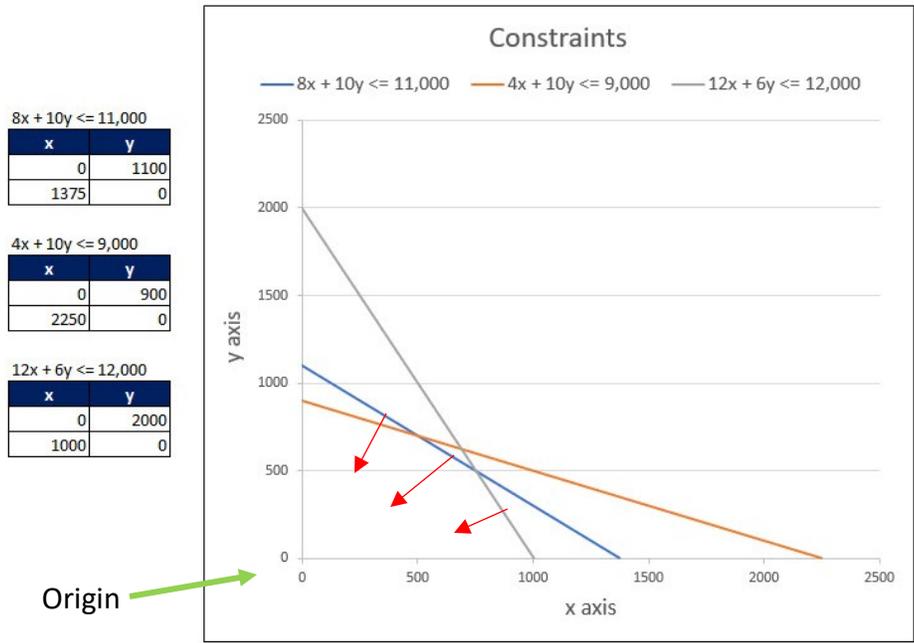


Figure 1

Since all the constraints are less-than-or-equal-to constraints, the direction we look towards that contains the solution points are the coordinates (0, 0), called the origin.

The convex shape that satisfies all three constraint lines (also called the intercepts) is called the feasible region (in blue in Figure 2).

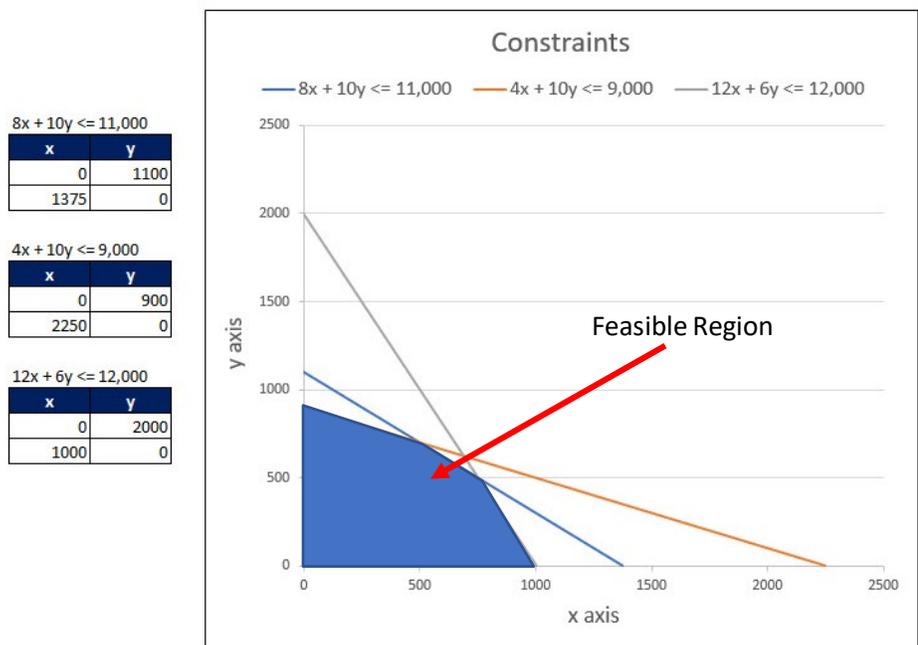


Figure 2

The feasible region indicates the area where there are an infinite number of solutions to the problem. Because this is an optimization problem, however, we are looking for the *optimal* solution.

In linear programming, the optimal solution falls on the extreme or corner points, the location where the constraint lines intercept on the outer-most edge of the feasible region. The example below shows there are five corner points and one of these points contain the optimal solution.

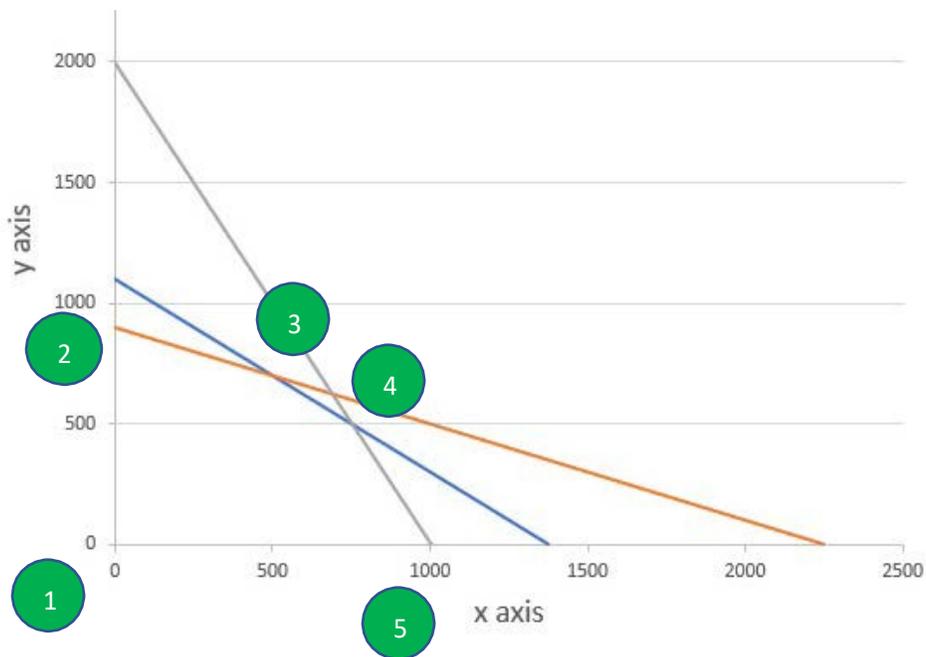


Figure 3

Figure 3 shows the coordinates for corner point 1 are $(0, 0)$. This denotes zero profits so this cannot be the optimal solution.

We know from Figure 1 the coordinates for corner point 2 are $(0, 900)$. We also know the coordinates for corner point 5 are $(1,000, 0)$.

We can guess the coordinates for corner points 3 and 4 by eyeballing, however, this usually doesn't yield a precise measurement.

The coordinates $(0, 0)$ indicate the company has made zero products - no product A or product B. We can next add a profit line since the company is looking to make a profit from its products produced. See the dotted line in Figure 4. Once the profit line moves outside the feasible region, the solution to the business problem is not profitable - the solution is not feasible or optimal.

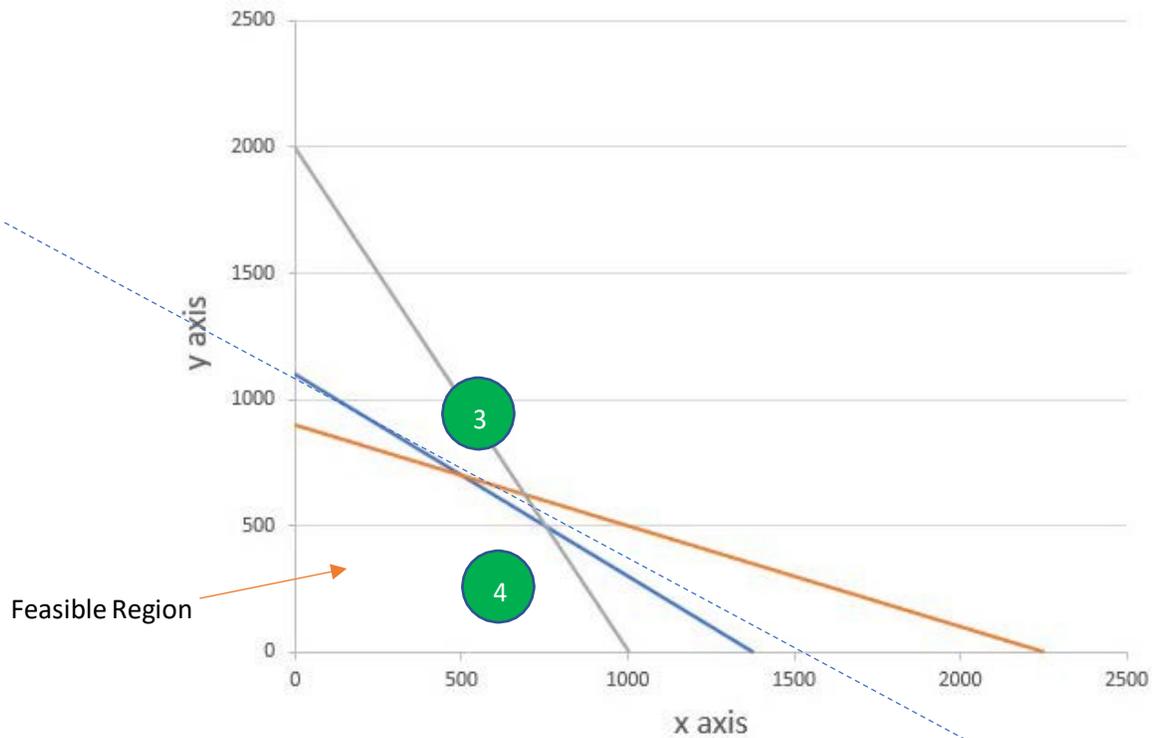


Figure 4

Figure 4 shows the last corner point the dotted profit line touches before exiting the feasible region is corner point 3. This indicates corner point 3 contains the optimal solution. Now we will find its coordinates.

The intercept for corner point 3 falls on the following constraint lines:

$$8x + 10y \leq 11,000 \text{ and } 4x + 10y \leq 9,000$$

Since the corner point falls on the *extreme edge* of the feasible region, I can change the inequality \leq to an equality $=$. We now get:

$$8x + 10y = 11,000$$

$$4x + 10y = 9,000$$

The equal signs tell us the left side of the equation is equal to the right side. When applying the same math operation to both sides of the equal sign, the equality will remain intact regardless of the math operation used to solve for x and y .

If I multiply the left and right side of the bottom equation by -1 , we get:

$$-1(4x + 10y = 9,000).$$

Now solving for x:

$$8x + 10y = 11,000$$

$$\underline{-4x - 10y = -9,000}$$

$$4x = 2,000$$

$$X = 500$$

We can substitute $x = 500$ into any of the two equations above to solve for y . Let's choose the top equation.

$$8(500) + 10y = 11,000$$

$$4000 + 10y = 11,000$$

Subtracting 4,000 from both sides.

$$10y = 7,000$$

$$y = 700$$

Going back to the problem in paragraph 4, the optimal product combination the company should produce is 500 units of product A and 700 units of product B.

Let's plug $x = 500$ and $y = 700$ into the objective function $z = 4(x) + 8(y)$.

$$\text{Profit (z)} = 4(500) + 8(700)$$

$$\$2,000 + \$5,600 = \$7,600$$

The optimal profit solution for product mix A & B is \$7,600.

In summary, this mathematical model can be computed faster using linear programming software Microsoft Solver, Lindo and Lingo. However, knowing how to build your business model algebraically and seeing graphically how the constraints limit the objective function will give you a deeper understanding when using linear programming software.

Because Solver is a free add-on for Excel and Microsoft's programs are more ubiquitous, I would recommend starting with Solver. Lindo and Lingo can be downloaded for free, however, with restrictions.

Linear programming, the graphical method, has its limitations. There are only two decision variables, the x and y coordinates. Most complex companies will have hundreds of decision variables and a similar high number of constraints.

Your next step will be to build your mathematical model in Excel using more than two decision variables. Your success will depend on how well you translate real-world business challenges and goals into an appropriate model with limited abstractions.